§2.3 Differential Formulas

- Basic Formulas: Notation: $f'(x) = \frac{df}{dx}$.

☆ Power function: For any n, $\boxed{(x^n)' = n \cdot x^{n-1}}$. In particularly, $(c)'=0$, $(x)'=1$.

eg.1. $(\sqrt[3]{x})' = (x^{\frac{1}{3}})' \overset{n=\frac{1}{3}}{=\!=\!=} \frac{1}{3} \cdot x^{\frac{1}{3}-1} = \boxed{\frac{1}{3} \cdot x^{-\frac{2}{3}}}$

$(\frac{1}{x})' = (x^{-1})' \overset{n=-1}{=\!=\!=} (-1) \cdot x^{-1-1} = \boxed{(-1) \cdot x^{-2} = \frac{-1}{x^2}}$

- Derivative Rules:

Sum/Difference: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$.

Constant multiple: $(c \cdot f(x))' = c \cdot (f(x))'$ for any constant c.

☆ Product: $[f(x) \cdot g(x)]' = f'(x) g(x) + f(x) \cdot g'(x)$. or $\boxed{(f \cdot g)' = f'g + f \cdot g'}$

☆ Quotient: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$ or $\boxed{\left(\frac{f}{g}\right)' = \frac{f' g - f \cdot g'}{g^2}}$

eg.2. Let $f(x) = 5 - \frac{3}{\sqrt{x}} + 2 \cdot x^{3.5}$. Compute $f'(x)$.

Solution: $f'(x) = 5' - \left(\frac{3}{\sqrt{x}}\right)' + (2 \cdot x^{3.5})'$

$= 0 - 3 \cdot (x^{-\frac{1}{2}})' + 2 \cdot (x^{3.5})'$

Hint: $\frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$

$= 0 - 3 \cdot (-\frac{1}{2}) \cdot x^{-\frac{1}{2}-1} + 2 \cdot 3.5 \cdot x^{3.5-1} = \boxed{\frac{3}{2} \cdot x^{-\frac{3}{2}} + 7 \cdot x^{2.5}}$

eg.3. (5.16). Let $g(x) = (\frac{1}{x^5} - 2x)(\frac{1}{\sqrt{x}} + \pi)$. Compute $g'(x)$.

Solution: $g'(x) = \left[\frac{1}{x^5} - 2x\right]' (\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x) \cdot (\frac{1}{\sqrt{x}} + \pi)'$

$= (x^{-5} - 2x)' \cdot (\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x) \cdot (x^{-\frac{1}{2}} + \pi)'$

$= \boxed{(-5 x^{-6} - 2) \cdot (\frac{1}{\sqrt{x}} + \pi) + (\frac{1}{x^5} - 2x) \cdot (-\frac{1}{2} x^{-\frac{3}{2}} + 0)}$

eg4. Let $f(x)=\frac{2}{x+1}$. Find the tangent line of $f(x)$ at $(3,\frac{1}{2})$.

Recall, the slope of the tangent line at $x=3$ equals the derivative at $x=3$.

i.e. We need to compute $f'(3)$ (or $\left.\frac{df}{dx}\right|_{x=3}$)

Solution: $f'(x)=\frac{2'(x+1)-2(x+1)'}{(x+1)^2}$ (quotient rule). $2'=0$, $(x+1)'=x'+1'=1+0$

$$=\frac{0-2}{(x+1)^2}=\frac{-2}{(x+1)^2} \Rightarrow f'(3)=\frac{-2}{(3+1)^2}=\frac{-2}{16}=-\frac{1}{8}$$

Therefore, the tangent formula:

$$\boxed{y-\frac{1}{2}=\left(-\frac{1}{8}\right)\cdot(x-3)}$$

Higher order derivatives:

The derivative function of $f'(x)$ is called the second derivative of $f(x)$, denoted

$$\boxed{f''(x)}=(f'(x))'=\frac{d^2f}{dx^2}=\frac{d}{dx}\left(\frac{df}{dx}\right)$$

In the same way, we can define third / fourth ... order derivative as $f'''(x)=[f''(x)]'$, ...

n-th order derivative is also denoted as $f^{(n)}(x)$, eg. $f^{(4)}(x)=f''''(x)=[f'''(x)]'$

Three important physical functions: Displacement $s(t)$, Velocity $V(t)$, Acceleration $a(t)$.

$\boxed{V(t)=s'(t),\quad a(t)=V'(t)}$. i.e. $a(t)$ is the second order derivative of $s(t)$, i.e. $a(t)=s''(t)$.

e.g.5. A particle moves according $s(t)=t^3-6t^2+5$, $t\geq 0$.
(F16). (a) Find the velocity at time t.
(b) What is the acceleration after 6 seconds?

Solution: (a) $V(t)=s'(t)=(t^3-6\cdot t^2+5)'=3\cdot t^2-6\cdot 2\cdot t+0=\boxed{3t^2-12t}$

(b) $a(t)=(V(t))'=(3t^2-12t)'=3\cdot 2t-12=\boxed{6t-12}$

so $a(6)=\left.(6t-12)\right|_{t=6}=6\cdot 6-12=\boxed{24}$ ft/s^2

§2.4. Trigonometric Derivatives.

Key formulas: • $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$ (θ can be replaced by any other variable

• $(\sin x)'=\cos x$, $(\cos x)'=-\sin x$, $(\tan x)'=\sec^2 x$, $(\sec x)'=\sec x\cdot\tan x$.

Rmk: $(\csc(x))'=-\cot(x)\cdot\csc(x)$, $(\cot(x))'=-\csc^2(x)$ are not required.

eg.1. Let $f(x)=2\sin x-x\cdot\cos x$. Find $f'(x)$, $f''(x)$ and $f''(\frac{\pi}{2})$.

Solution: $f'(x)=(2\sin x)'-(x\cdot\cos x)'$

$=2\cdot\cos x-[x'\cdot\cos x+x\cdot(\cos x)']$

$=2\cos x-[1\cdot\cos x+x\cdot(-\sin x)]=2\cos x-\cos x+x\cdot\sin x=\boxed{\cos x+x\cdot\sin x}$

$f''(x)=(\cos x)'+(x\cdot\sin x)'$

$=-\sin x+x'\cdot\sin x+x(\sin x)'=-\sin x+\sin x+x\cdot\cos x=\boxed{x\cos x}$

$f''(\frac{\pi}{2})=\frac{\pi}{2}\cos\frac{\pi}{2}=\boxed{\frac{\pi}{2}\cdot 0}=0$. Caution: $\sin 0=\cos\frac{\pi}{2}=0$, $\sin\frac{\pi}{2}=\cos 0=1$.

eg 2 Let $h(\theta)=\frac{2\theta}{\sec\theta}$. Compute $h'(\theta)$ and $h'(\frac{\pi}{4})$

Solution: $h'(\theta)\xlongequal[\text{rule}]{\text{quotient}}\frac{(2\theta)'\cdot\sec\theta-(2\theta)\cdot(\sec\theta)'}{(\sec\theta)^2}=\frac{2\cdot\sec\theta-2\theta\cdot\sec\theta\cdot\tan\theta}{\sec^2\theta}=\boxed{\frac{2-2\theta\cdot\tan\theta}{\sec\theta}}$

$h'(\frac{\pi}{4})=\frac{2-2\cdot\frac{\pi}{4}\cdot\tan\frac{\pi}{4}}{\sec\frac{\pi}{4}}=\frac{2-2\cdot\frac{\pi}{4}\cdot 1}{\sqrt{2}}=\boxed{\frac{2-\frac{\pi}{2}}{\sqrt{2}}}$. Hint: $\tan\frac{\pi}{4}=1$, $\sec\frac{\pi}{4}=\sqrt{2}$

eg 3. Use the trig-relation $\tan x=\frac{\sin x}{\cos x}$ and quotient rule to derive the formula for $\tan' x$.

$(\tan x)'=\left(\frac{\sin x}{\cos x}\right)'=\frac{(\sin x)'\cdot\cos x-\sin x\cdot(\cos x)'}{\cos^2 x}=\frac{\cos x\cdot\cos x-\sin x\cdot(-\sin x)}{\cos^2 x}$

$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\boxed{\sec^2 x}$

• Application of the limits $\lim\limits_{\blacksquare\to 0}\dfrac{\sin\blacksquare}{\blacksquare}=1$, $\lim\limits_{\blacktriangle\to 0}\dfrac{1-\cos\blacktriangle}{\blacktriangle}=0$.

eg.4. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin(3x)}{2x}$. Rmk: We need the exact form $\dfrac{\sin(\ )}{(\ )}$ SAME.

Solution: $\lim\limits_{x\to 0}\dfrac{\sin(3x)}{2x}\cdot\dfrac{3x}{3x}=\lim\limits_{x\to 0}\dfrac{\sin(3x)}{3x}\cdot\dfrac{3x}{2x}=\lim\limits_{x\to 0}\dfrac{\sin(3x)}{3x}\cdot\lim\limits_{x\to 0}\dfrac{3\cancel{x}}{2\cancel{x}}$

$$=1\cdot\frac{3}{2}=\boxed{\frac{3}{2}}$$

~~eg.5. Evaluate~~ $\cancel{\lim\limits_{h\to 0}\dfrac{3h}{\cos h-1}}$

~~solution:~~ $\cancel{=\lim\limits_{h\to 0}\dfrac{3}{-1}\cdot\dfrac{h}{1-\cos h}=\lim\limits_{h\to 0}\dfrac{3}{-1}\cdot\dfrac{1}{\frac{1-\cos h}{h}}=\dfrac{3}{-1}\cdot\dfrac{1}{1}=\boxed{-3}}$

☆ eg.6. Find the limit $\lim\limits_{x\to 0}\dfrac{\sin(x^2+6x)}{x}$.

(F16). $=\lim\limits_{x\to 0}\dfrac{\sin(x^2+6x)}{x}\cdot\dfrac{x^2+6x}{x^2+6x}=\lim\limits_{x\to 0}\left[\dfrac{\sin(x^2+6x)}{x^2+6x}\right]\cdot\left[\dfrac{x^2+6x}{x}\right]$

$$=\lim_{x\to 0}\left[\frac{\sin(x^2+6x)}{x^2+6x}\right]\cdot\lim_{x\to 0}(x+6)=1\cdot 6=\boxed{6}$$

☆ eg.7 Find the limit $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x^2-1}$

(S17)

Solution: $=\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x^2-1}\cdot\dfrac{x-1}{x-1}=\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x-1}\cdot\dfrac{x-1}{x^2-1}$ Notice that $x^2-1=(x+1)(x-1)$

$$=\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\cdot\lim_{x\to 1}\frac{\cancel{x-1}}{(x+1)\cdot\cancel{(x-1)}}$$

$$=[1]\cdot\left[\frac{1}{1+1}\right]=\boxed{\frac{1}{2}}$$